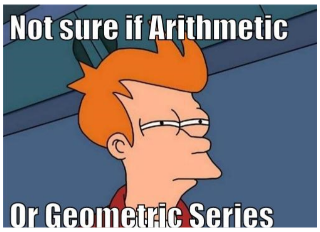
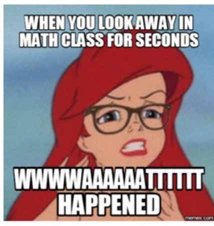
Sequences & Series: Arithmetic & Geometric





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This is a long worksheet to cater for students that want extra practice. If you want a shortcut, but still be sure to cover one of each type then follow the pink highlighted questions.

- **Higher level students** should be able to do all questions to be sure to get a 7
- **Standard level students** should be able to do questions 1-56 and 63-77 to be sure to get a grade 7

The sections 'with … ' should only be done once a student has already covered those topics in their course.

Note: There is no sigma notation or recurrence relations in this sheet. These topics are on a separate worksheet.

Quick Reminders

Notation and Definitions

n	u_n	S_n
n = term number This is always a whole number since it represents a certain term. For example: $n = 1$ means look for 1st term $n = 3$ means look for 3rd term	$u_n = n^{th}$ term This is a specific/particular term for any given n. For example: u_1 tells us what the 1st term is u_3 tells us what the 3rd term is Watch out: Sometimes courses use letters other than u (like a)	This is the sum of n terms. In other words, the sum up until a point. For example: S_2 tells us the sum of the first 2 terms S_3 tells us the sum of the first 3 terms

Let's demonstrate this with 2 examples:

If given a sequence: $3, 5, 7, 9, \ldots, 401$	If given the n^{th} term: $u_n = 2n^2 - 4n + 1$
All we know from this sequence is that: $u_1 = 3$ $u_2 = 5$ $u_3 = 7$ $u_4 = 9$ $u_n = 401$ (we don't know which term this sequence is so we call it u_n) We can also work out the sums of some terms $S_1 = 3$ $S_2 = 3+5 = 8$ $S_3 = 3+5+7 = 15$ $S_4 = 3+5+7+9 = 24$ We would like to be able to do things like: • Know what the 102nd term is straight away or any other high term in the sequence. Yes, we could write more terms down just by spotting the pattern of adding 2 each time and work this out, but this would be a bit tedious. • Know how many terms the sequence has, just by knowing that the last term is • Find the sum of a large number of terms • Find a rule to generate any term in the sequence (find any term we want, not just the ones that we are given). See the example on the right when we are given a rule.	We can calculate the terms here by plugging in values of n. The question always ready gives us the rule, so plugging values of n in generates the sequence for us. 1st term: Let $n = 1$ $u_1 = 2n^2 - 4n + 1 = 2(1)^2 - 4(1) + 1 = -1$ 2nd term: Let $n = 2$ $u_2 = 2n^2 - 4n + 1 = 2(2)^2 - 4(2) + 1 = 1$ 3rd term: Let $n = 3$ $u_3 = 2n^2 - 4n + 1 = 2(3)^2 - 4(3) + 1 = 7$ 4th term: Let $n = 4$ $u_4 = 2n^2 - 4n + 1 = 2(4)^2 - 4(4) + 1 = 17$ Our sequence looks like $\{-1, 1, 7, 17, \ldots\}$ Sum of the first 4 terms: $S_4 = -1 + 1 + 7 + 17 = 24$ It is great when we are given the rule. We can generate any term we want quickly. We are not always given the rule, so it is not as simple as always just being able to plug numbers in. We need a method to be able to generate the rule ourselves from a given sequence.

The limitations mentioned above mean we need a systematic way to deal with sequences. Fortunately, there are only 2 types of sequences that we have to deal (arithmetic and geometric). First of all, let's distinguish between these two types of sequences.

- **Arithmetic** sequences are where you **add** or **subtract** the SAME number which we call the **common difference**
- **Geometric** sequences are where you **multiply** or **divide** by the SAME number each time which we call the **common ratio**.

For these two sequences we use the following notations:

- a = first term. This is the same as saying u_1.
- d = common difference. This is the same number that you add or subtract by each time in an arithmetic sequence. We subtract any term by its previous term to get this.
- r = common ratio. This is the same number that you multiply or divide by each time in a geometric sequence. We divide any term by its previous term to get this.

Let's look at an example of each.

Consider the sequence $1, 3, 5, 7, 9, \ldots, 54$	Consider the sequence $2, 4, 8, 16, \ldots, 512$
We add 2 each time, so the series must be arithmetic • $d = 3 - 1 = 2$. Note: we could have also done $5 - 3 = 2$ or $= 7 - 5 = 2$ etc • $u_1 = a = 1$ (first term) • $u_n = 54$ (last term) • $S_3 = 1 + 3 + 5 = 9$	We multiply by 2 each time, so the series must be geometric • $r = \frac{4}{2} = 2$ Note: or we could have done $\frac{8}{4} = 2$ or $\frac{16}{8} = 2$ etc • $u_1 = a = 2$ (first term) • $u_n = 15$ (last term) • $S_3 = 2 + 4 + 8 = 14$

Method

Step 1: Ask yourself whether the series is arithmetic or geometric
Step 2: Ask yourself whether you know the first term is (a)
Step 3: Ask yourself whether you know what the common different d is (if arithmetic) or what the common ratio r is (if geometric)
Step 4: Plug into the formulae u_n and/or s_n (these formulae are given in the table on the page below)
Step 5: Solving using algebra knowledge (this could be an easy equation or a simultaneous equation)

Once you have understood all the above, it just a question of using either the formula or the definition.

- **Using the formulae** (your first thought should always be how can I use the formulae)

Arithmetic sequence	Geometric sequence
➢ $u_n = a + (n-1)d$ where d is the difference between any term and its previous term a is the first term ➢ $S_n = \frac{n}{2}[2a + (n-1)d]$ ➢ $S_n = \frac{n}{2}[a + l]$ where l is the last term Notice how there are two formulae for s_n. We use the second one when we know the last time.	➢ $u_n = ar^{n-1}$ where r is $\frac{\text{any term}}{\text{previous term}}$ a is the first term ➢ $S_n = \frac{a(1-r^n)}{1-r}$ or $\frac{a(r^n-1)}{r-1}$ Note you can use either of these and I personally always use the first , but it makes life easier for students (in terms of avoiding negatives) if we use the first formula when r <1 and the second when r >1 ➢ $S_\infty = \frac{a}{1-r}$ This only can be used if the sequence converges (if $-1 < r < 1$)

How do we use the formulae? We either
- ✓ have the values of all the unknowns and can find what we want easily by plugging into the formula
- ✓ don't have the values of the knowns in which case we have to work backwards to solve for the unknown (harder questions will involve solving simultaneous equations)

Some examples:

Work out the nth term of 4,9,14,19, ... i. Work out the 95th term	In an arithmetic sequence $u_1 = 2$, $u_{20} = 78$ and $u_n = 3710$ Find d and n	Consider the arithmetic sequence 3,9,15, … ,1353 Find the sum of the sequence	The 40th term of an arithmetic sequence is 106 and the sum of the first forty terms is 1900. Find the first term and the common difference
4 ,9, 14 ,19,... first term $= a = 4$ common difference $= d = 9-4 = 5$ We use the formula $u_n = a + (n-1)d$ Let's replace a and d in the formula $4 + (n-1)5$ Simplify $4 + 5n - 5$ $5n - 1$ i. Now that we have the rule we can work out any term 95th term means $n = 95$ $5(95) - 1 = 474$	We use the formula $u_n = a + (n-1)d$ We already know $a = u_1 = 2$ We also know $u_{20} = 78$. We use the formula for the lef- hand side: $2 + (20-1)d = 78$ $2 + (20-1)d = 78$ $76 = 19d$ $d = 4$ We also know $u_n = 3710$. We use the formula for the left-hand side: $2 + (n-1)(4) = 3710$ $2 + 4n - 4 = 3710$ $3712 = 4n$ $n = 928$	first term $= a = 3$ common difference $= d = 9-3 = 6$ We know the final term is 1353. We need to know what term number this corresponds to. We say $u_n = 1353$ We use the formula $u_n = a + (n-1)d$ $a + (n-1)d = 1353$ Now let's replace a and d $3 + (n-1)(6) = 1353$ $3 + 6n - 6 = 1353$ $6n = 1356$ $n = 226$ Now that we know we have 226 terms we can find the sum of the sequence using the formulas for s_n where $n = 226$ $S_{226} = \frac{226}{2}[2(3) + (226-1)6]$ $= 153228$	Here we don't know a and d, so we will need to work backwards to find them by using the formula The fortieth term of an arithmetic sequence is 106 tells us that $u_{40} = 106$ The sum of the first 40 terms of an arithmetic sequence tells us that $s_{40} = 1900$ Let's turn this into the u_n and s_n formulae $u_{40} = 106$ — The formula tells us $a + 39d = 106$ $s_{40} = 1900$ — The formula tells us $\frac{40}{2}[2a + 39(d)] = 1900$ So, we have two equations: $a + 39d = 106$ ① $20[2a + 39(d)] = 1900$ ② Simplifying both we get $a + 39d = 106$ $2a + 39d = 950$ Solve simultaneously $a = 12, d = -3$

The same method applies for geometric series, just with a different formula obviously.

- **Using the definition**:

Occasionally we are given the sequences in terms of unknowns and using the formula becomes problematic since there are too many unknowns.
- If a,b,c,d,... is an arithmetic sequence then we can build and solve the equation $b - a = c - b$
- If a,b,c,d,... is a geometric sequence then we can build and solve the equation $\frac{b}{a} = \frac{d}{c}$
- $u_n = s_n - s_{n-1}$ is useful if we are given the sum in terms of n and want to find u_n

The first three terms of an arithmetic sequence are $12 - p, 2p, 4p - 5$. Find the value of p and hence the first term and common difference of the series	The first three terms of a geometric sequence are $a, a + 14, 9a$. Find the value of a.
What do we know about an arithmetic sequence? If we subtract the successive terms, they must give the same answer. $2p - (12 - p) = (4p - 5) - 2p$ $2p - 12 + p = 4p - 5 - 2p$ $p = 7$ Put this back into sequence $12 - 7, 2(7), 4(7) - 5$ 5,14,23 $a = 5, d = 14 - 5 = 9$ Note: we could have also done $12 - p - 2p = 2p - (4p - 5)$ etc.	What do we know about and geometric sequence? If we divide the successive terms, they must give the same answer. $\frac{a}{a+14} = \frac{a+14}{9a}$ $9a^2 = (a+14)^2$ $9a^2 = a^2 + 28a + 196$ $8a^2 - 28a - 196 = 0$ $2a^2 - 7a - 49 = 0$ $(2a+7)(a-7) = 0$ So, we actually have two solutions, $a = 7$ or $-\frac{7}{2}$.

Harder Types Of Questions

You will often see arithmetic and geometric series topics combined with other topics such as logs, trig, binomial expansion and sectors, so make sure your knowledge in these areas is also good.

Also, watch out for harder types of questions such as:

- **Greatest possible sum:** solve $u_n > 0$ to find out how many positive terms and then take the sum of these terms
- **Inequalities:** especially with logs. The sign swaps when multiplying or dividing by a negative. Remember log (number less than one) is negative
- **Solving geometric** series **simultaneously**: They are harder to solve, and the quickest method is to just divide successive terms in order to get unknowns to cancel
- Working out **how many years**:
  - ✓ if from beginning to year n to end of year m we do (m−n)+1 OR m−(n−1) i.e. we want to include n so don't take it away in the second formula.
  - ✓ if from beginning to year n to beginning of year m its $m - n$
- Questions with **lots of words**: We try and get these out of the words and into symbols u_n ands s_n as soon as possible and then follow our usual procedure of:

  Step 1: Ask yourself whether arithmetic or geometric

  Step 2: Ask yourself what a is

  Step 3: Ask yourself what d is (if arithmetic) or what r is (if geometric)

  Step 4: Plug into u_n and/or s_n

1 Bronze

1.1 Without Formulae

1) The nth term of a sequence is $2n - 3$
   i. Find the 42^{nd} term
   ii. Find the term number which is 1455

2) The nth term of a sequence is $(n + 3)(n - 4)$
   i. Find the 20^{th} term
   ii. Find the term number that has the value of 78

3) Find u_1, u_2, u_3 and u_{10} of the sequence $u_n = 3n + 2$

4) Find the value of n for which u_n has the given value, $u_n = 2n - 4, u_n = 24$

5) The nth term of a sequence is $3{+}4n$.
   i. Find the first 3 terms of the sequence.
   ii. Find the value of n for which u_n has the value 27
   iii. Find S_5

6) Find S_5 where $u_n = 2n + 5$

7) The first three terms of an arithmetic sequence are $12 - p, 2p, 4p - 5$ respectively, where p is a constant. Find the value of p and hence the first term and common difference of the series

8) The first three terms of a geometric sequence are $a, a + 14, 9a$. Find the value of a

9) Consider the geometric sequence $x - 3$, $x + 1$, $2x + 8$. When $x = 5$, the series is geometric
   i. Write down the first three terms
   ii. Find the common ratio
   iii. Find the other value of x for which the sequence is geometric
   iv. For this value of x, find the common ratio

1.2 Using Formulae

10) Work out the nth term of $4,9,14,19, \ldots$

11) The first three terms of an arithmetic sequence are $36,40,44$
   i. Find the common difference
   ii. Find u_n
   iii. Show that $s_n = 2n^2 + 34n$
   iv. Hence write down the value of s_{14}

12) Consider the arithmetic sequence $2,5,8,11,\ldots$
    i. Find u_{101}
    ii. Find the value of n so that $u_n = 152$

13) An arithmetic sequence is given by $5,8,11,\ldots$
    i. Write down the value of d
    ii. Find u_{100}
    iii. Find s_{100}
    iv. Given that $u_n = 1502$, find the value of n

14) Let s_n, be the sum of the first n terms of the arithmetic series $2 + 4 + 6 + \cdots$
    i. Find s_4
    ii. Find s_{100}

15) In an arithmetic sequence $u_1 = 2$ and $u_2 = 8$
    i. Find d
    ii. Find u_{20}
    iii. Find s_{20}

16) The first term of a geometric progression is 12 and the second term is -6. Find
    i. The tenth term
    ii. The sum to infinity

17) In an arithmetic sequence $u_1 = 2$, $u_{20} = 78$ and $u_n = 3710$
    i. Find d
    ii. Find the value of n

18) In an arithmetic sequence the first term is 5 and the fourth term is 40, find the second term

19) In an arithmetic sequence the first term is -7 and the sum of the first 20 terms is 620
    i. Find the common difference
    ii. Find the value of the $78th$ term

2 Silver

2.1 Without Formulae

20) The sum of the first n terms of a series is given by is $s_n = 2n^2 - n$, where $n \in \mathbb{Z}^+$ Find the first three terms of the series and find an expression for the n^{th} term of the sequence, giving your answer in terms of n.

21) The sum s_n of the n terms of an arithmetic progression is given by $s_n = 32n - n^2$. Find the first term and common difference.

22) The sum of the first n terms of an arithmetic sequence is given by $s_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m, and u_{32}, are consecutive terms of a geometric sequence. Find m.

2.2 Using Formulae

2.2.1 Finding n

23) Consider the geometric sequence $5, 10, \dots, 1280$. Find the number of terms in the sequence.

24) Consider the arithmetic sequence $3, 9, 15, \dots, 1353$. Find the sum of the sequence

25) Find the sum of all the multiples of 3 between 100 and 1000

2.2.2 Simultaneous Equations

26) The fourth term of an arithmetic sequence is 3 and the sum of the first 6 terms is 27. Find the first term and the common difference

27) The 40th term of an arithmetic sequence is 106 and the sum of the first forty terms is 1900. Find the first term and the common difference

28) The fourth term of a geometric is 10 and the seventh term of the series is 80. For this series, find
    i. The common ratio
    ii. The first term
    iii. The sum of the first 20 terms, giving your answer to the nearest whole number

29) The sum to infinity of a geometric sequence is 16 and the sum of the first four terms is 15. Find the possible first terms and common ratios.

30) The sum of the first 16 terms of an arithmetic sequence is 212 and the fifth term is 8. Find the first term and common difference.

31) The sum of the first three terms of a geometric sequence is 62.755, and the sum of the infinite sequence is 440. Find the common ratio.

32) The sum of the first 2 terms of a geometric sequence is 6 and the sum of the first 4 terms of a geometric sequence is 30. Find the first term and common ratio

2.2.3 With Inequalities

33) Find the **greatest possible sum** of the arithmetic sequence $85, 78, 71$

34) The third term of an arithmetic sequence is 1407 and the $10th$ term is 1183.
   i. Find the first term and common difference of the sequence
   ii. Calculate the number of **positive terms** of the sequence

35) An arithmetic sequence has first term 1000 and common difference of -6. The sum of the first n terms of this sequence is **negative**. Find the **least value** of n

36) The 5th term of an arithmetic series is 16 and the 10th term is 30
   i. Find the first term and common difference
   ii. How many terms of the series are needed for the sum of the series to **exceed** 1000?

37) Find the least number of terms required for the sum of $4 + 9 + 14 + 19 + \cdots$ to **exceed** 2000.

2.2.4 Worded Questions

38) Carol starts a new job on a salary of £$20{,}000$. She is given an annual wage rise of £500 at the end of every year until she reaches her maximum salary of £$25{,}000$. Find the total she earns (assuming no other rises)
   i. In the first 10 years
   ii. Over 15 years
   iii. State one reason why this may be an unsuitable model

39) A company extracted 4500 tonnes of minerals from a mine during 2018. The mass of the mineral which the company expects to extract in each subsequent year is modelled to decrease by 2% each year.
   i. Find the total mass of the mineral which the company expects to extract from 2018 to 2040 inclusive, giving your answer to 3 significant figures
   ii. Find the mass of the mineral which the company expects to extract during 2040, giving your answer to 3 significant figures

   The costs of extracting the mineral each year are assumed to be
   £800 per tonne for the first 1500 tones
   £600 per tonne for any amount in excess of 1500 tonnes
   The expected cost of extracting the mineral from 2018 to 2040 inclusive is £x million
   iii. Find the value of x

40) A small company which makes batteries for electric cars has a 10-year plan for growth
- In year 1 the company will make $2{,}600$ batteries
- In year 10 the company aims to make $12{,}000$ batteries

In order to calculate the number of batteries it will need to make each year, form year 2 to year 9, the company considers two models, Model A and Model B

In Model A the number of batteries male will increase by the same **number** each year
   i. Using Model A, determine the number of batteries the company will make in year 2

In model B the number of batteries will increase by the same **percentage** each year
   ii. Using model B, determine the number of batteries the company will make in year 2. Give you answer to the nearest 10 batteries.

Sam calculates the total number of batteries make from year 1 to year 10 inclusive using each of the 2 models
   iii. Calculate the difference between the two totals, giving your answer to the nearest 100 batteries

41) Lewis played a game of space invaders. He scored points for each spaceship that he captured. Lewis scored 140 points for capturing his first spaceship. He scored 160 points for capturing his second spaceship, 180 points for capturing his third spaceship, and so on. The number of points scored for capturing each successive spaceship formed an arithmetic sequence.
   i. Find the number of points that Lewis scored for capturing his $20th$ spaceship
   ii. Find the total number of points Lewis scored for capturing his first 20 spaceships

   Sian played an adventure game. She scored points for each dragon that she captured. The number of points that Sian scored for capturing each successive dragon formed an arithmetic sequence. Sian captured n dragons and the total number of points that she scored for capturing all n dragons was 8500. Given that Sian scored 300 points for capturing her first dragon and then 700 points for capturing her n^{th} dragon,
   iii. find the value of n

2.2.4.1 *Being Careful with difference between years*

42) **Finding n**: Jess started work 20 years ago. In year 1 her annual salary was $£17{,}000$. Her annual salary increased by $£1{,}500$ each year, so that her annual salary in year 2 was $£18{,}500$, in year 3 it was $£20{,}000$ and so on, forming an arithmetic sequence. This continued until she reached her maximum annual salary of $£32{,}000$ in year k. Her annual salary then remained at $£32{,}000$.
   i. Find the value of the constant k
   ii. Calculate the total amount that Jess has earned in the 20 years

43) On John's 10th birthday he received the first of an annual birthday gift of money from his uncle. This first gift was $£60$ and on each subsequent birthday the gift was $£15$ more than the year before. The amounts of these gifts form an arithmetic sequence
   i. Show that, after his 12th birthday, the total of these gifts was $£225$
   ii. Find the amount that John received from his uncle as a birthday gift on his 18th birthday.
   iii. Find the total of these birthday gifts that John had received from his uncle up to and including his 21st birthday.

   When John had received n of these birthday gifts, the total money that he had received from these gifts was $£3375$
   iv. Show that $n^2 + 7n = 25 \times 18$
   v. Find the value of n, when he had received $£3375$ in total, and so determine John's age at this time

44) A company, which is making 200 mobile phones each week, plans to increase its production. The number of mobile phones produced is to be increased by 20 each week from 200 in week 1 to 220 in week 2, to 240 in week 3 and so on, until it is producing 600 in week N
   i. Find the value of N

   The company then plans to continue to make 600 mobile phones each week
   ii. Find the total number of mobile phones that will be made in the first 52 weeks starting from and including week 1

45) A company, which is making 140 bicycles each week, plans to increase its production. The number of bicycles produced is to be increased by d each week from 140 in week 1 to $140 + d$ in week 2, to $140 + 2d$ in week 3 and so on, until it is producing 206 in week 12.
   i. Find the value of d

   After week 12 the company plans to continue to make 206 bicycles each week
   ii. Find the total number of bicycles that would be made in the first 52 weeks starting from and including week 1

46) Portable telephones are first sold in the country Cellmania in 1990. During 1990, the number of units sold is 160. In 1991, the number of units sold is 240 and in 1992, the number of units sold is 360. In 1993 it was noticed that the annual sales formed a geometric sequence with first term 160, the $2nd$ and $3rd$ terms being 240 and 360 respectively
   i. How many units are sold during 2002?
   ii. In what year does the number of units sold first exceed 5000?
   iii. Between 1990 and 1992 the total number of units sold is 760. What is the total number of units sold between 1990 and 2002?

3 Gold

3.1 Without Formulae

47) The first three terms of a geometric sequence are $\frac{k-3}{2}, \frac{2k+3}{4}$ *and* $\frac{12k+3}{8}$
   i. Show that k satisfies the equation $8k^2 - 45k - 18 = 0$
   ii. Find the first 5 terms of the sequence
   Hint: Write out the sequence after finding k and the term and common ratio

48) Find two distinct numbers p and q such that $p, q, 10$ are in arithmetic progression and $q, p, 10$ are in geometric progression.

3.2 Using Formulae

3.2.1 Simultaneous Equations

49) In a geometric series, the sum of the second and third term is -12 and the sum of the third and fourth term is -36. Find the common ratio.

50) The sum of the first two terms of a geometric progression is 8 and the sum of the next two terms is 2. Find the possible values of the common ratio and for each one. Find the first term of the progression.

51) The sum of the 1st and 2nd terms of a geometric progression is 50 and the sum of the 2nd and 3rd terms is 30. Find the sum to infinity.

52) In a geometric sequence, the first term is $3a$ and the common ratio is r. In another geometric sequence the first term is a and the common ratio is $-2r$. The sums to infinites are equal for both sequences. Find the common ratio.

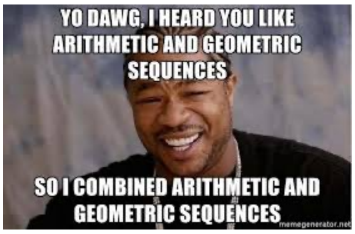
53) A geometric series is such that s_{10} is four times S_5. Find the exact value of r.

54) In a geometric sequence, the fourth term is 8 times the first term. The sum of the first 10 terms is 2557.5. Find the 10th term of this sequence.

55) The first term of an infinite geometric series exceeds the second term by 9. The sum of the series is 81. Find the common ratio.

56) An arithmetic progression is such that the eighth term is three times the third term. Show that the sum of the first eight terms is four times the sum of the first four terms.

3.2.2 Arithmetic and Geometric Together In One Question



57) The 1st, 2nd and 3rd terms of a geometric progression are the 1st, 9th and 21st terms respectively of an arithmetic progression. The 1st term of each progression is 8 and the common ratio of the geometric is r, where $r \neq 1$. Find
  i. The value of r
  ii. The 4th term of each sequence

58) The 1st term of an arithmetic progression is 2. The 1st, 2nd and 5th terms of this progression are the first 3 terms of a geometric progression respectively. Find
  i. The common difference of the arithmetic progression
  ii. The common ratio of the geometric progression

59) A geometric sequence $u_1, u_2, u_3, \ldots$has $u_1 = 27$ and a sum to infinity of $\frac{81}{2}$
  i. Find the common ratio of the geometric sequence.

An arithmetic sequence $v_1, v_2, v_3, \ldots$ is such that $v_2 = u_2$ and $v_4 = u_4$
  ii. Find the greatest value of N such that $\sum_{n=1}^{N} v_n > 0$

60) An arithmetic sequence $\{u_n : n \in \mathbb{Z}^+\}$ has first term $u_1 = 1.6$ and a common difference $d = 1.5$. The geometric sequence $\{v_n : n \in \mathbb{Z}^+\}$ has first term $v_1 = 3$ and common ratio $r = 1.2$
  i. Find an expression for $u_n - v_n$ in terms of n
  ii. Determine the set of values of n for which $u_n > v_n$
  iii. Determine the greatest value of $u_n - v_n$. Giving your answer correct to four significant figures

61) The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence. The arithmetic sequence has first term a and non-zero common difference d.
  i. Show that $d = \frac{a}{2}$
  ii. The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200. Find the least value of n for which this occurs.

62) A sum of the first n terms of an arithmetic sequence $\{u_n\}$ is given by formula $s_n = 4n^2 - 2n$. Three terms of sequence u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m

3.2.3 Worded Questions

63) Xin has been given a 14-day training schedule by her coach. Xin will run for A minutes on day 1, where A is a constant. She will then increase her running time by $(d + 1)$ minutes each day, where d is a constant.
  i. Show that on day 14, Xin will run for $(A + 13d + 13)$ minutes.

Yi has also been given a 14-day training schedule by her coach. Yi will run for $(A - 13)$ minutes on day 1. She will then increase her running time by $(2d - 1)$ minutes each day. Given that Yi and Xin will run for the same length of time on day 14,
  ii. find the value of d.

Given that Xin runs for a total time of 784 minutes over the 14 days,
  iii. find the value of A

64) A store begins to stock a new range of DVD players and achieves sales of £1500 of these products during the first month. In a model it is assumed that sales will decrease by £x in each subsequent month, so that sales of £$(1500 - x)$ and £$(1500 - 2x)$ will be achieved in the second and third months respectively. Given that sales total £8100 during the first six months, use the model to
  i. find the value of x
  ii. find the expected value of sales in the eighth month
  iii. show that the expected total of sales in pounds during the first n months is given by $kn(51 - n)$, where k is an integer to be found.
  iv. Explain why this model cannot be valid over a long period of time

65) A farmer has a pay scheme to keep fruit pickers working throughout the 30 day season. He pays £a for their first day, £$(a + d)$ for their second day, £$(a + 2d)$ for their third day, and so on, thus increasing the daily payment by £d for each extra day they work. A picker who works for all 30 days will earn £40.75 on the final day.
   i. Use this information to form an equation in a and d.

A picker who works for all 30 days will earn a total of £1005
   ii. Show that $15(a + 40.75) = 1005$
   iii. Hence find the value of a and the value of d

66) Shelim starts his new job on a salary of £$14\,000$. He will receive a rise of £1500 a year for each full year that he works, so that he will have a salary of £$15\,500$ in year 2, a salary of £$17\,000$ in year 3 and so on. When Shelim's salary reaches £$26\,000$, he will receive no more rises. His salary will remain at £$26\,000$.
   i. Show that Shelim will have a salary of £$26\,000$ in year 9
   ii. Find the total amount that Shelim will earn in his job in the first 9 years

Anna starts her new job at the same time as Shelim on a salary of £A. She receives a rise of £1000 a year for each full year that she works, so that she has a salary of £$(A + 1000)$ in year 2, £$(A + 2000)$ in year 3 and so on. The maximum salary for her job, which is reached in year 10, is also £$26\,000$.
   iii. Find the difference in the total amount earned by Shelim and Anna in the first 10 years

3.2.4 With Logs

67) An arithmetic sequence has first term $\ln a$ and common difference $\ln 3$. The 13th term of the sequence is $8ln9$. Find the value of a.

68) The first two terms of an infinite geometric sequence in order are $2\,log_2 x,\ log_2 x$ where $x > 0$.
   i. Find r
   ii. Show that the sum of the infinite sequence is $4\log_2 x$

The first three terms of an arithmetic sequence in order are

$$log_2 x,\ log_2 \frac{x}{2},\ log_2 \frac{x}{4} \text{ where } x > 0$$

   iii. Find d, giving your answer as an integer

Let S_{12} be the sum of the first 12 terms of the arithmetic sequence
   iv. Show that $S_{12} = 12\log_2 x - 66$
   v. Given that S_{12} is equal to half the sum of the infinite geometric sequence, find x, giving your answer in the form 2^p, where $p \in \mathbb{Q}$

3.2.5 With Inequalities and Logs

69) How many terms are needed for the sum of the geometric series $3 + 6 + 12 + 24 + \cdots$ to **exceed** $100{,}000$

70) The second and third terms of a geometric series are 192 and 144 respectively. Find the smallest value of n for which the sum of the first n terms of the series exceeds 1000

71) The first term of a geometric series is 120. The sum to infinity of the series is 480. The sum of the n terms of the series is greater than 300. Calculate the smallest possible values of n.

72) In a geometric series $u_1 = \frac{1}{81}$ and $u_4 = \frac{1}{3}$
   i. Find the value of r
   ii. Find the **smallest value** of n which $s_n > 40$

73) The first three term of a geometric sequence are $u_1 = 0.64, u_2 = 1.6$, and $u_3 = 4$.
   i. Find the value of r
   ii. Find the value of s_6
   iii. Find the **least value** of n such that $s_n > 75000$

74) Carlos saves money every year. The first year he saves £100. Each year he increases the amount he saves by 10%. After how many years do Carlos's savings first exceed £1000 (excluding any interest he has earned)

75) A geometric series has first term 5 and common ratio $\frac{4}{5}$. Given that the sum of the k terms of the series is greater than 24.95
   i. Show that $k > \frac{\log 0.002}{\log 0.8}$
   ii. Find the smallest possible value of k

76) The first term of a geometric series is 20 and the common ratio is $\frac{7}{8}$. The sum of the N terms of the series is S_N. Find the smallest value of N, for which $S_\infty - S_N < 0.5$

77) The adult population of a town is $25{,}000$ at the end of year 1. A model predicts that the adult population of the town will increase by 3% each year, forming a geometric sequence
   i. Show that the predicted adult population at the end of year 2 is $25{,}750$
   ii. Write down the common ratio of the geometric sequence

   The model predicts that year N will be the first year in which the adult population of the town exceeds $40{,}000$
   iii. Show that $(N-1)\log 1.03 > \log 1.6$
   iv. Find the value of N

   At the end of each year, each member of the adult population of the town will give £1 to a charity fund. Assuming the population model,
   v. Find the total amount that will be given to the charity fund for the 10 years from the end of year 1 to the end of year 10, giving your answer to the nearest £1000.

3.2.6 With Trig

78) Find the set of values of θ, such that $-\pi < \theta < \pi$, for which the geometric progression $1 + 2\cos^2\theta + 4\cos^4\theta + 8\cos^6\theta + \cdots$ has a sum to infinity.
   i. Show that for this value of θ, the sum to infinity of the progression is $-\sec 2\theta$.

79) Given that the first three terms of a geometric series are

$$12\cos\theta \qquad 5 + 2\sin\theta \qquad \text{and} \qquad 6\tan\theta$$

   i. Show that

$$4\sin^2\theta - 52\sin\theta + 25 = 0$$

   Given that θ is an obtuse angle measures in radians,
   ii. Solve the equation in part i. to find the exact value of θ
   iii. Show that the sum to infinity of the series can be expressed in the form

$$k(1-\sqrt{3})$$

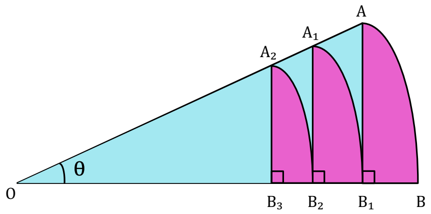
   where k is a constant to be found

3.2.7 With Sectors

80) A circular plank is cut into 12 sectors whose areas are in arithmetic progression. If the area of the largest sector is twice that of the smallest, find the angle in terms of π between the straight edges of the smallest sector.

81) A circular disc is cut into 12 sectors whose areas are in arithmetic progression. If the angle of the largest sector is twice that of the smallest sector, find the size of the angle of the smallest sector

82) The diagram shows a sector AOB of a circle of radius 1 with centre O, where angle $AOB = \theta$. The lines (AB_1), (A_1B_2), (A_2B_3) are perpendicular to OB. (A_1B_1), (A_2B_2) are all arcs of circles with centre O



Calculate the sum to infinity of the arc lengths $AB + A_1B_1 + A_2B_2 + A_3B_3 + \cdots$

4 Diamond

4.1 Using Formulae

4.1.1 Simultaneous Equations

83) An arithmetic sequence has first term 15 and second term 19. Another arithmetic sequence has first term 420 and second term 415. The sum of each sequence is the same for a particular n. What is n?

84) Find in terms of k, the 50^{th} term of the arithmetic sequence $(2k+1), (4k+4), (6k+7)$. ... Give your answer in its simplest form

85) A geometric series has first term $\mathrm{b}^2 - 13$, common ratio $\frac{1}{\mathrm{b}}$ and sum to infinity -6. Find all possible values of the common ratio.

86) In the arithmetic series $k + 2k + 3k + \cdots + 100$, k is a positive integer and k is a factor of 100.
    i. Find, in terms of k, an expression for the number of terms in the series
    ii. Show that the sum of the series is $50 + \frac{5000}{\mathrm{k}}$

87) The first term in an arithmetic series is $5t + 3$, where t is a positive integer. The last term is $17t + 11$ and the common difference is 4. Show that the sum of the series is divisible by 12 when, and only when, t is odd.

4.1.2 Arithmetic and Geometric Together in One Question

88) The sums of the terms of a sequence follow the pattern $S_1 = 1 + k$, $S_2 = 5 + 3\mathrm{k}$, $S_3 = 12 + 7\mathrm{k}$, $S_4 = 22 + 15\mathrm{k}, \ldots$, where $\mathrm{k} \in \mathbb{Z}$.
    i. Given that $\mathrm{u}_1 = 1 + \mathrm{k}$, find $\mathrm{u}_2, \mathrm{u}_3$, and u_4 $(ans\ 4 + 2k, 7 + 4k, 10 + 8k)$
    ii. Find a general expression for u_n $(ans = 3n - 2 + 2^{n-1}k)$

89) The first 3 terms of a geometric series are also the 3rd, 14th and 58th terms of an arithmetic series. Given the sum of the first 3 terms of the arithmetic series is 24, and the sum of the first 5 terms is 55, find the sum of the first 5 terms of the geometric sequence

90) The sum of the first three numbers in an arithmetic sequence is 24. If the first number is decreased by 1 and the second number is decreased by 2, then the third number and the two new numbers are in geometric sequence. Find all possible sets of three numbers which are in the arithmetic sequence.

91) The n^{th} term of a geometric progression is denoted by g_n and the n^{th} term of an arithmetic progression is denoted by a_n. It is given that $\mathrm{a}_1 = \mathrm{g}_1 = 1 + \sqrt{5}$, $\mathrm{g}_3 = \mathrm{a}_2$ and $\mathrm{g}_4 + \mathrm{a}_3$ =0. Given also that the geometric progression is convergent, show that it's sum to infinity is $4 + 2\sqrt{5}$.

92) Each of the terms of an arithmetic series is added to the corresponding term of a geometric series, forming a new series with first term $\frac{3}{8}$ and second term $\frac{13}{16}$. The common difference of the arithmetic series is four times

as large as the first term of the geometric series. The common ratio of the geometric series is twice as large as the first term of the arithmetic series. Determine the possible value of the first term of the geometric series.

4.2 With Binomial Expansion

93) Given that the coefficients of x, x^2 and x^4 in the expansion of $(1+kx)^n$, where $n \geq 4$ and k is a positive constant, are the consecutive terms of a geometric sequence, show that $k = \frac{6(n-1)}{(n-2)(n-3)}$.

4.3 With Logs

94) The first terms of an arithmetic sequence are $\frac{1}{log_2 x}, \frac{1}{log_8 x}, \frac{1}{log_{32} x}, \frac{1}{log_{128} x}, ...$ Find x if the sum of the first 20 terms of the sequence is equal to 100

Hint: d simplified is $\frac{2}{log_2 x}$, and $a = \frac{1}{log_2 x}$. Plug into s_n